

The Product and Quotient Rules



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Rule 5: The Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In other words:

The derivative of f times g is **the first times the derivative of the second plus the second times the derivative of the first.**

It is helpful to learning the rule if you are able to repeat the words in red as you are using the rule.

Example of the Product Rule:

Find the derivative of $f(x) = (2x^2 - 1)(3x + 4)$

Since the function is written as a product of two functions you could use the product rule or multiply it out and use the basic rules of differentiation. We'll look at it BOTH ways.

Find the derivative of $f(x) = (2x^2 - 1)(3x + 4)$.

1. The Product Rule

$$\text{(first)} \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the second} \end{array} \right) + \text{(second)} \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the first} \end{array} \right)$$

$$f'(x) = (2x^2 - 1) \cdot \frac{d}{dx}(3x + 4) + (3x + 4) \cdot \frac{d}{dx}(2x^2 - 1)$$

$$f'(x) = (2x^2 - 1) \cdot 3 + (3x + 4) \cdot 4x$$

$$f'(x) = 6x^2 - 3 + 12x^2 + 16x$$

$$f'(x) = 18x^2 + 16x - 3$$

2. Same derivative by expanding and using the Power Rule.

$$f(x) = (2x^2 - 1)(3x + 4)$$

$$f(x) = 6x^3 + 8x^2 - 3x - 4$$

$$f'(x) = 18x^2 + 16x - 3$$

Notice in the first example, finding the derivative of the function $f(x) = (2x^2 - 1)(3x + 4)$, the derivative could be found two different ways. Whether you use the Product Rule or rewrite the function by multiplying and find the derivative using the Power Rule, the result or the derivative,

$$f'(x) = 18x^2 + 16x - 3, \text{ was the same.}$$

Example 2 of the Product Rule:

Find $f'(x)$ for $f(x) = x^3(\sqrt{x} - 1)$

First rewrite the radical in exponential form.

$$f(x) = x^3 \left(x^{\frac{1}{2}} - 1 \right)$$

Now use the Product Rule.

$$f'(x) = (\text{first}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the second} \end{array} \right) + (\text{second}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the first} \end{array} \right)$$

$$f'(x) = x^3 \frac{d}{dx} \left(x^{\frac{1}{2}} - 1 \right) + \left(x^{\frac{1}{2}} - 1 \right) \frac{d}{dx} (x^3)$$

Example 2 continued...

$$f'(x) = x^3 \frac{d}{dx} \left(x^{\frac{1}{2}} - 1 \right) + \left(x^{\frac{1}{2}} - 1 \right) \frac{d}{dx} (x^3)$$

$$f'(x) = x^3 \left(\frac{1}{2} \cdot x^{-\frac{1}{2}} - 0 \right) + \left(x^{\frac{1}{2}} - 1 \right) 3 \cdot x^2$$

$$f'(x) = \frac{1}{2} x^{\frac{5}{2}} + 3x^{\frac{5}{2}} - 3x^2$$

Recall when you are multiplying
the same base you add the exponents.

$$f'(x) = \frac{7}{2} x^{\frac{5}{2}} - 3x^2$$

*The derivative could also be found by performing the multiplication and then finding the derivative.

Rule 6: The Quotient Rule

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx} (f(x)) - f(x) \frac{d}{dx} (g(x))}{[g(x)]^2}$$

This rule may look overwhelming with the functions but it is easy to learn if you can repeat these words: **The derivative of a quotient is the bottom times the derivative of the top minus the top times the derivative of the bottom over the bottom squared.**

$$\left(\begin{array}{l} \text{Derivative of} \\ \text{a quotient} \end{array} \right) = \frac{(\text{bottom}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the top} \end{array} \right) - (\text{top}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the bottom} \end{array} \right)}{(\text{bottom})^2}$$

Here it is written as a fraction:

$$\left(\begin{array}{l} \text{Derivative of} \\ \text{a quotient} \end{array} \right) = \frac{(\text{bottom}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the top} \end{array} \right) - (\text{top}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the bottom} \end{array} \right)}{(\text{bottom})^2}$$

Keep in mind that when we say “top” and “bottom” we are referring to the numerator and denominator of the original function respectively. Another important thing to remember with this rule is the order in the numerator of the derivative!!

Example of the Quotient Rule: Find $\frac{d}{dx} \left(\frac{x^2 - 3x - 1}{x} \right)$

First we will find the derivative by using The Quotient Rule

$$\left(\begin{array}{l} \text{Derivative of} \\ \text{a quotient} \end{array} \right) = \frac{(\text{bottom}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the top} \end{array} \right) - (\text{top}) \cdot \left(\begin{array}{l} \text{derivative of} \\ \text{the bottom} \end{array} \right)}{(\text{bottom})^2}$$

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2 - 3x - 1}{x} \right) &= \frac{x \frac{d}{dx} (x^2 - 3x - 1) - (x^2 - 3x - 1) \frac{d}{dx} (x)}{x^2} \\ &= \frac{x \cdot (2x - 3) - (x^2 - 3x - 1) \cdot 1}{x^2} \\ &= \frac{2x^2 - 3x - x^2 + 3x + 1}{x^2} \\ &= \frac{x^2 + 1}{x^2} \end{aligned}$$

Another way to do the same problem is to do the division first and then use the power rule.

$$\frac{x^2 - 3x - 1}{x} = \frac{x^2}{x} - \frac{3x}{x} - \frac{1}{x} = x - 3 - x^{-1}$$

Now find the derivative.

$$\frac{d}{dx}(x - 3 - x^{-1}) = 1 - 0 - (-1) \cdot x^{-2} = 1 + x^{-2} = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$$

Again, notice there is more than one method you could use to find the derivative.

Example 2: Find the derivative of $f(x) = \frac{x}{x^2 + 1}$

For this quotient doing the division first would require polynomial long division and is not going to eliminate the need to use the Quotient Rule. So you will want to just use the Quotient Rule.

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1) \cdot \frac{d}{dx}(x) - (x) \cdot \frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2} \\ &= \frac{(x^2 + 1) \cdot 1 - (x) \cdot 2x}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \\ &= \frac{-x^2 + 1}{(x^2 + 1)^2} \end{aligned}$$

The Product Rule and the Quotient Rule can be very helpful when you are finding derivatives. In the next section you may be using them in conjunction with another rule called the Chain Rule.

It is important that you learn the two rules correctly!! Being able to say the rules in words will help you find derivatives correctly. Remember that the order in the Quotient Rule is important.